Formula One Race Strategy

McLaren Racing Limited

Sports Technology

The Royal Academy of Engineering

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INTRODUCTION

Figure 1: A Vodafone McLaren Mercedes driven by Lewis Hamilton

McLaren Racing, the company behind Vodafone McLaren Mercedes, operates in the highly competitive and technological environment of Formula One. McLaren Racing comprises a multitude of administrative and engineering departments, ranging from vehicle design and aerodynamics, to materials science and the paint shop. As one of the most successful teams in the history of Formula One, McLaren has won more Grand Prix than any other Constructor since it entered the sport in 1966.

IMPORTANCE OF EXEMPLAR IN REAL LIFE

Every F1 team must decide how much fuel their cars will start each race with, and the laps on which they will stop to refuel and change tyres. In the sport, this is known as Race Strategy. A wise choice can get a driver to the finish quicker, helping him gain more points and maybe even a world championship.

SCENARIO

McLaren have entered a race of length 50 laps. We plan to stop once on lap 20 to re-fuel. Our fuel consumption is 3kg a lap. Each kg of fuel we have in our fuel tank at the start of a lap makes us 0.03sec/lap slower than our fastest possible lap time of 100.045sec/lap. When we stop to re-fuel, we take 20 seconds (plus the time to put the fuel in) longer to complete that lap. It takes 0.5 sec to put in one lap's worth of fuel. How long will it take us to complete the race if we stop only once?

MATHEMATICAL MODEL

According to the scenario, we have:

$$\begin{aligned}\text{Total Race Time} &= \text{Time to reach first stop}\\ &\quad + \text{Time for first stop} \quad \ldots(1)\\ &\quad + \text{Time to end of race}\end{aligned}$$

We will solve this problem in three steps as follows:

Step 1: How long will it take to reach the first pit stop?

From the scenario, we have the following information:

Fuel Consumption	C	=	3 kg/lap
How much slower our lap time is for every kg of fuel on board (also called the "weight effect")	E	=	0.03 sec/(lap kg)
Time to complete a lap with 1 lap of fuel on board	t_1	=	100.045 sec

Using this, we can calculate how much slower the car goes for every lap's worth of fuel we have on board. We call this the Fuel Laps Weight Effect (W) and this is calculated as follows:

$$\begin{aligned}W &= \text{Fuel Laps Weight Effect}\\ &= \text{Fuel Consumption} \times \text{Weight Effect}\\ &= 3 \times 0.03 = 0.09 \text{ sec/(lap lap of fuel)}\end{aligned}$$

The extra time taken to complete a lap when we have fuel on board can be calculated as follows:

$$\begin{aligned}&\text{Extra time taken to complete lap due to fuel on board}\\ &= \text{Average number of laps of fuel on board}\\ &\quad \times \text{Fuel Laps Weight Effect}\end{aligned} \quad \ldots (2)$$

We call this the Fuel Load Effect and the value of this is listed in the 4th column of Table 1 for each of the first twenty laps, e.g. for the first lap, we have:

$$\begin{aligned}&\text{Extra time taken to complete lap due to fuel on board}\\ &= \left(\frac{20+19}{2}\right) \times 0.09 = 1.775 \text{ sec}\end{aligned}$$

Next, we need to calculate the perfect laptime (t_0). This is the time it would take to complete a lap if the car had no fuel on board (but was magically still able to run normally!):

$$\begin{aligned}t_0 &= \text{Time to complete a lap with 1 lap of fuel on board}\\ &\quad - (\text{Average no. of laps of fuel on board} \times W)\\ &= t_1 - \left[\left(\frac{1+0}{2}\right) \times W\right]\\ &= 100.045 - (0.5 \times 0.09) = 100 \text{ sec}\end{aligned}$$

Finally, we can calculate the time to complete a lap as follows:

$$\text{Lap Time} = \text{Fuel Load Effect} + t_0 \quad \ldots (3)$$

This is calculated and listed in the 5th column of Table 1.

Lap	Laps of fuel on board at start of lap	Laps of fuel on board at end of lap	Fuel Load Effect (sec)	Lap Time (sec)
1	20	19	1.755	101.755
2	19	18	1.665	101.665
3	18	17	1.575	101.575
4	17	16	1.485	101.485
5	16	15	1.395	101.395
6	15	14	1.305	101.305
7	14	13	1.215	101.215
8	13	12	1.125	101.125
9	12	11	1.035	101.035
10	11	10	0.945	100.945
11	10	9	0.855	100.855
12	9	8	0.765	100.765
13	8	7	0.675	100.675
14	7	6	0.585	100.585
15	6	5	0.495	100.495
16	5	4	0.405	100.405
17	4	3	0.315	100.315
18	3	2	0.225	100.225
19	2	1	0.135	100.135
20	1	0	0.045	100.045

Table 1: Calculations for time taken to complete each lap

When we add all the values in column 5 of Table 1, we get the total time taken to reach the pit stop on lap 20. This is 2018 seconds, which is approximately 33.6 minutes. This is known as the time to complete the *first stint*.

Step 2: How long will the pit stop on lap 20 take?

Again, we have the following data from the scenario:

Total number of laps	L_{end}	=	50
Stop lap	L_2	=	20
Time to add one lap of fuel	t_f	=	0.5 sec
Extra time to complete a lap with a pit stop but without refuelling	t_p	=	20 sec
Fuel consumption	C	=	3 kg/lap

Using this data, we calculate:

$$\begin{aligned} &\text{Number of laps of fuel to add at pit stop} \\ &= \text{Total number of laps} - \text{laps already done} \\ &= L_{end} - L_2 = 50 - 20 = 30 \text{ laps of fuel} \end{aligned}$$

$$\begin{aligned} &\text{Fuel flow time during stop} \\ &= \text{Laps of fuel to add} \times \text{Time to add one lap of fuel} \\ &= 30 \times 0.5 = 15 \text{ sec} \end{aligned}$$

$$\begin{aligned} &\text{Extra time for a lap with a pit stop} \\ &= \text{Fuel flow time} \\ &\quad + \text{Extra time to complete a lap with a pitstop} \\ &\quad\ \text{but without refuelling} \\ &= 15 + t_p = 15 + 20 = 35 \text{ sec} \end{aligned}$$

Hence, it will take 35 seconds more to complete the lap with the pit stop on lap 20.

Step 3: How long will it take to complete the rest of the race after the pit stop on lap 20?

This can be calculated in a similar way to step 2. However, this time we begin with 30 laps of fuel and have 30 laps to complete. The table for this *second stint* is shown below:

Lap	Laps of fuel on board at start of lap	Laps of fuel on board at end of lap	Fuel Load Effect (sec)	Lap Time (sec)
21	30	29	2.655	102.655
22	29	28	2.565	102.565
23	28	27	2.475	102.475
24	27	26	2.385	102.385
25	26	25	2.295	102.295
26	25	24	2.205	102.205
27	24	23	2.115	102.115
28	23	22	2.025	102.025
29	22	21	1.935	101.935
30	21	20	1.845	101.845
31	20	19	1.755	101.755
32	19	18	1.665	101.665
33	18	17	1.575	101.575
34	17	16	1.485	101.485
35	16	15	1.395	101.395
36	15	14	1.305	101.305
37	14	13	1.215	101.215
38	13	12	1.125	101.125
39	12	11	1.035	101.035
40	11	10	0.945	100.945
41	10	9	0.855	100.855
42	9	8	0.765	100.765
43	8	7	0.675	100.675
44	7	6	0.585	100.585
45	6	5	0.495	100.495
46	5	4	0.405	100.405
47	4	3	0.315	100.315
48	3	2	0.225	100.225
49	2	1	0.135	100.135
50	1	0	0.045	100.045

Table 2: Calculations for time taken to complete each lap

When we add all the values in column 5 of Table 2 we get the total time taken to reach the end of the race from lap 20. This is 3040.5 seconds.

Hence, if we are planning to stop only once during the race on lap 20, it will take approximately 85 minutes to complete the race.

$$\begin{aligned}\text{Total Race Time} &= 2018 + 35 + 3040.5\\ &= 5093.5\ \text{sec}\\ &= 84.9\ \text{minutes}\end{aligned}$$

ALTERNATIVE MATHEMATICAL MODEL

Let's review what we've just done. We've calculated the time it took to complete each lap, then added all the lap times up to give the total race time. Essentially, we are summing the area under graphs like the one shown in Figure 2 (below).

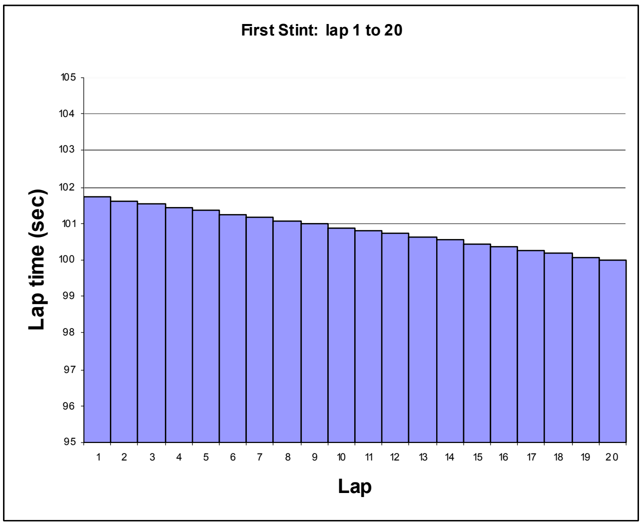

Figure 2: Lap times in first stint

This can be done much more easily using integration rather than summing values in a table with the help of a calculator.

Using Integration, we have:

$$\begin{aligned}&\text{Total Race Time}\\ &= \text{Integral of lap times over first stint}\\ &\quad + \text{Time for first stop}\\ &\quad + \text{Integral of lap times over second stint}\end{aligned} \quad \ldots (4)$$

The formula for the lap time on any given lap is similar to the one we used to calculated column 5 in our tables above:

$$\text{Lap Time} = \left[t_0 + (L_2 - l)W\right] \ \ldots (5)$$

where l is the number of laps completed. The fact that l is now continuous over a lap automatically averages the fuel load over the lap. Check this yourself: if you integrate it over any specific lap it will give the correct lap time.

Integrating equation (5) over a stint, we get:

$$\text{Stint Time} = \int_{L_1}^{L_2} \left[t_0 + (L_2 - l)W\right] dl \quad \ldots (6)$$

The solution to this integral is:

$$\text{Stint Time} = \left[t_0 l + L_2 W l - \frac{W l^2}{2}\right]_{L_1}^{L_2} \quad \ldots (7)$$

For the first stint, we have the following data from the scenario:

t_0	=	100 sec
L_1 and L_2	=	0 and 20 laps respectively
W	=	0.09 sec/(lap lap of fuel)

Substituting these values, we get:

First Stint Time

$$= (100 \times 20) + (20 \times 0.09 \times 20) - \left(\frac{0.09 \times 20 \times 20}{2}\right)$$

$$= 2018\ \text{sec}$$

This is the same answer as that calculated in step 1 but with a lot *less* work!

EXTENSION ACTIVITY – 1:

Using the solution to the general integral formula for calculating stint time given in equation (7), calculate the time for the second stint and verify it with the one achieved in step 3.

EXTENSION ACTIVITY – 2:

a) Would we have a longer or shorter race time if we didn't stop at all?

b) Which lap should we stop on if we want to make our race time as short as possible?

c) Why is this not the halfway lap?

Did you know?

The only reason stopping for fuel is quicker than starting with enough fuel to finish the race is because the average mass of a car which re-fuels during a race is lower than when it doesn't. When the car is lighter it takes less time to complete a lap. The reason behind this is as follows:

Essentially, Newton's second law (F= ma) applies to the car as it travels around the track. From this, we can see that if mass goes up but forces remain unchanged then accelerations must reduce- and lower accelerations mean increased lap times.

In reality, it is slightly more complicated- all of the forces don't stay the same when fuel load (and hence car mass) changes. Tyre frictional forces ($F = \mu N$, where μ is the coefficient of friction and N is the normal force) change substantially, but Aerodynamic forces stay largely the same (lift and drag) and these are large. However, the net effect is that a heavy car (i.e. one that is full of fuel) takes longer to get around the track.

WHERE TO FIND MORE

1. *Basic Engineering Mathematics*, John Bird, 2007, published by Elsevier Ltd.
2. *Engineering Mathematics*, Fifth Edition, John Bird, 2007, published by Elsevier Ltd.
3. Visit www.autosport.com or any website on *Formula One Car Racing* for more information.

Photograph Awaited

William Mulholland, Vehicle Dynamics Engineer, McLaren Racing Limited

William holds an MEng from Cambridge University.

"As a Formula one engineer, I use mathematics and physics every day to try and help McLaren win more F1 races. If you like maths problems, and get a buzz from solving ones like in this exemplar, then you should think about going on to study engineering at university and coming to work at McLaren".

INFORMATION FOR TEACHERS

The teachers should have some knowledge of

- Simple calculations using empirical formulae and a calculator
- Tabulating the findings in each step
- Representing data graphically
- Using derivatives to find optimum points
- Integration
- Area under the graph

TOPICS COVERED FROM "MATHEMATICS FOR ENGINEERING"

- Topic 1: Mathematical Models in Engineering
- Topic 4: Functions
- Topic 6: Differentiation and Integration

LEARNING OUTCOMES

- LO 01: Understand the idea of mathematical modelling
- LO 04: Understand the mathematical structure of a range of functions and be familiar with their graphs
- LO 06: Know how to use differentiation and integration in the context of engineering analysis and problem solving
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

ASSESSMENT CRITERIA

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 4.1: Identify and describe functions and their graphs
- AC 6.2: Use derivatives to classify stationary points of a function of one variable
- AC 6.3: Find definite and indefinite integrals of functions
- AC 6.4: Use integration to find areas and volumes
- AC 8.1: Summarise a set of data
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

LINKS TO OTHER UNITS OF THE ADVANCED DIPLOMA IN ENGINEERING

- Unit-1: Investigating Engineering Business and the Environment
- Unit-4: Instrumentation and Control Engineering
- Unit-6: Investigating Modern Manufacturing Techniques used in Engineering
- Unit-7: Innovative Design and Enterprise
- Unit-8: Mathematical Techniques and Applications for Engineers
- Unit-9: Principles and Application of Engineering Science

Answers to Extension Activities

EA1:

Using equation (7):

$$\text{Stint Time} = \left[t_0 l + L_2 wl - \frac{wl^2}{2} \right]_{L_1}^{L_2}$$

with:

t_0	=	100 sec
L_1 and L_2	=	20 and 50 respectively
W	=	0.09 sec/(lap lap of fuel)

we get:

Second Stint Time

$$= \left[(100 \times 50) + (50 \times 0.09 \times 50) - \left(\frac{0.09 \times 50 \times 50}{2} \right) \right]$$

$$- \left[(100 \times 20) + (50 \times 0.09 \times 20) - \left(\frac{0.09 \times 20 \times 20}{2} \right) \right]$$

$$= 3040.5\,\text{sec}$$

EA2 (a):

Would we have a longer or shorter race time if we didn't stop at all?

For a non stop race we have only one stint. This means we can use equation 4 for the whole race.

t_0	=	100 sec
L_1 and L_2	=	0 and 50 respectively
W	=	0.09 sec/(lap lap of fuel)

Substituting these values, we get:

Total Race Time

$$= (100 \times 50) + (50 \times 0.09 \times 50) - \left(\frac{0.09 \times 50 \times 50}{2} \right)$$

$$= 5112.5\,\text{sec}$$

This is 19 seconds longer than it took the car that stopped once on lap 20. This is shown in Figure 3 below:

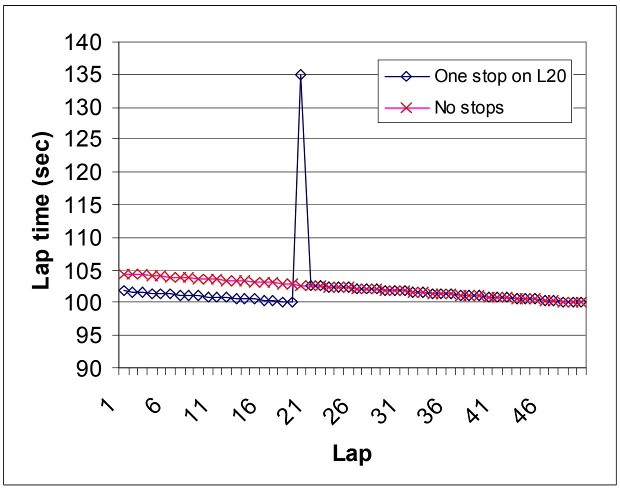

Figure 3: Race Lap times

EA2 (b):

Which lap should we stop on if we want to make our race time as short as possible?

First, let's assume only one stop. The equation for the time taken to complete the race with one stop is already given in equation (1). If we stop on Lap L_2, then using equation (7) for first and second stint together, we get from equation (1):

Total Race Time

$$= \left[t_0 l + L_2 Wl - \frac{Wl^2}{2} \right]_{L_1}^{L_2} + \left[(L_{end} - L_2) \times t_f + t_p \right]$$

$$+ \left[t_0 l + L_{end} Wl - \frac{Wl^2}{2} \right]_{L_2}^{L_{end}} \quad \dots (8)$$

where all the terms are same as defined earlier.

Expanding this with $L_1 = 0$ gives:

Total Race Time

$$= \left[t_0 L_2 + L_2 W L_2 - \frac{W L_2^2}{2} \right] + \left[(L_{end} - L_2) \times t_f + t_p \right]$$

$$+ \left[t_0 (L_{end} - L_2) + L_{end} W (L_{end} - L_2) - \frac{W (L_{end}^2 - L_2^2)}{2} \right] \quad \dots..(9)$$

Differentiating equation (9) with respect to L_2 and setting the LHS to zero, we can find the stop lap L_2 that gives the minimum value for race time:

$$0 = 2WL_2 - t_f - L_{end} W \quad \dots (10)$$

Solving for L_2 gives:

$$L_2 = \frac{L_{end}}{2} + \frac{t_f}{2W} \quad \dots (11)$$

Substituting the known values, we get:

$$L_2 = \frac{50}{2} + \frac{0.5}{2 \times 0.09} = 25 + 2.78 = 27.78$$

which we round up to the nearest lap: 28. So, if we plan to stop once, then stopping on lap 28 gives the quickest race time. You can check this by calculating the total race time for each of the stop laps between 20 and 30 using the method in part (a) and (b) of this extension. The answers to this are shown in Figure 4.

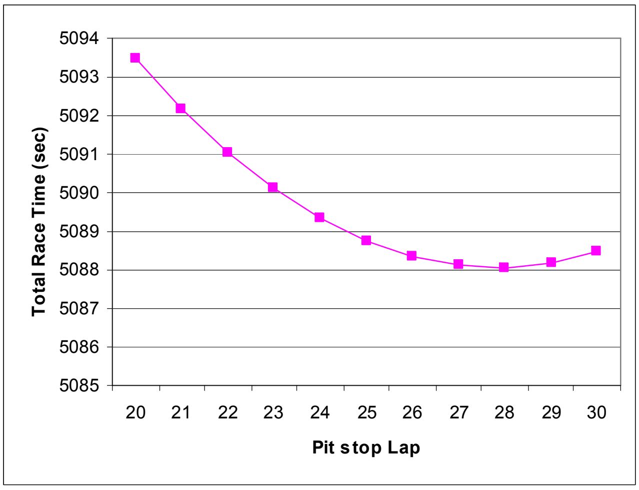

Figure 4: Total race times, varying first stop lap

The quickest race time is 5088.06 sec, stopping on lap 28. This is 5.44 sec quicker than one stop on lap 20.

EA2 (c):

Why is this not the halfway lap?

Equation (11) gives us the optimal stop lap. The first term on the RHS is the halfway point. We add a second term to this, which moves us away from the half way point. In this term, t_f is the time taken to add one lap of fuel, while W is the Fuel Lap Weight Effect. So, if it takes a longer time to add a lap of fuel (i.e. t_f is larger) then the second term will be greater and the optimal stop lap will be further away from the halfway point. Conversely, if carrying one lap of fuel adds more to our lap time (i.e. W is larger) then $\frac{t_f}{2W}$ will be smaller and our stop lap will be closer to half race distance.

This arrangement gives us a clue to the physical meaning of this relationship. It accounts for the fact that the fuel you begin the race with is put in the tank before the race starts (and so doesn't count toward total race time) whereas the time to add fuel during a pitstop counts toward total race time. So, if it takes a long time to fill up with a lap of fuel (i.e. t_f is large), then it is quicker to start with more fuel and go further into the race before stopping. This means you add less fuel during the stop. Of course, this only works if the lap time penalty for being heavy in the first stint is low (i.e. W is low).